

# THERMODYNAMICS OF A HOMOGENEOUS MIXTURE OF GAS, RADIATION, AND TURBULENCE

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## ABSTRACT

Subject to certain constraints, the equilibrium thermodynamics of a mixture of ionizing gas, blackbody radiation, and homogeneous turbulence is worked out in some detail. Specific heats and adiabatic exponents (Chandrasekhar’s gammas) are calculated. The adiabatic exponents show only a weak dependence on the turbulent pressure and the turbulent energy. This dependence is weakest under conditions of strong partial ionization of the gas, or when the relative pressure due to radiation is very high. Analytic expressions and numerical examples are both given, and possible astrophysical applications are briefly discussed.

*Subject headings:* atomic processes — equation of state — plasmas — turbulence

## 1. INTRODUCTION

Turbulence is a very common phenomenon in nature. Astrophysical systems that are strongly turbulent include convectively unstable planetary atmospheres, stellar envelopes and cores, stellar and galactic accretion disks and jets, and interstellar clouds, among many other objects in which fluid flows are of sufficiently low viscosity to be called turbulent. Very often, the turbulent velocities become so large in these objects that the turbulent pressure due to the Reynolds stresses must be expressly included in the equations of fluid motion.

In the simplest situation, consisting of a system in approximate hydrostatic equilibrium where one ignores all time derivatives, microscopic viscosities, buoyancy forces, rotation, and magnetic fields, the momentum equation becomes (Tennekes & Lumley 1972, pp. 32, 98)

$$\frac{1}{\rho} \frac{\partial}{\partial x_j} (p \delta_{ij} + \overline{\rho u_i u_j}) = -g_i . \quad (1)$$

Here  $u_i$  represents the turbulent velocity fluctuation,  $g_i = (0, 0, g)$  is the gravitational acceleration,  $\delta_{ij}$  is Kronecker’s delta,  $\rho$  is the mass density, and  $p$  is the sum of the thermal gas pressure  $P_{\text{gas}}$  and the radiation pressure  $P_{\text{rad}}$ . Taking a horizontal mean of equation (1) and considering the vertical direction only, we have

$$\frac{1}{\rho} \frac{d}{dz} (p + \overline{\rho w^2}) = -g . \quad (2)$$

If now we write  $\overline{w^2} = C v_{\text{turb}}^2$ , then

$$dP/dz = -g\rho , \quad (3)$$

where

$$P = P_{\text{gas}} + P_{\text{rad}} + P_{\text{turb}} \quad (4)$$

and

$$P_{\text{turb}} = C v_{\text{turb}}^2 \rho . \quad (5)$$

In the case of isotropic turbulence,  $C = 1/3$ . Note that equation (3) is just the familiar form of the equation of hydrostatic equilibrium.

Astrophysicists, however, need some sort of temporal treatment of the turbulent pressure in order to discuss stability problems. Their obvious choice is either to treat the turbulent pressure, properly, as part of the hydrodynamics

or to treat it, in some approximate fashion, as part of the thermodynamics. A hydrodynamical treatment is analytically impossible and numerically impractical (owing to the many different length and timescales involved). As a rough remedy, therefore, a thermodynamical treatment based on some idealized, simple phenomenological model would seem to be useful. We could assume, for example, a “gas” of turbulent eddies that move independently, are in thermal equilibrium, do not interact with the gas and radiation, and are neither created nor destroyed during the course of an adiabatic (rapid) perturbation. Within a sizable volume, which is nevertheless a small part of the overall astrophysical system and can be considered to be a closed subsystem, the turbulent eddies exhibit a spectrum of sizes; however, the larger, more energetic ones have longer lifetimes, rationalizing our assumption about their approximate integrity. Because large astrophysical systems like stars and interstellar clouds have long lives compared to the convective turnover times of turbulent eddies, the eddies may be assumed to be in some kind of local thermodynamic equilibrium among themselves and with the gas and radiation, even if they are created and destroyed rapidly. Of course, turbulent eddies are strongly correlated and interact nonlinearly on various timescales, as they cascade upward and downward in energy and produce Reynolds stresses (tensors) that are not conventional pressures (scalars). Therefore, we could alternatively dispense with any model, but simply fit the data from approximate numerical simulations of astrophysical turbulence to formal thermodynamic expressions for turbulent pressure and turbulent energy. Since the necessary simulations are not yet available and our physical picture of turbulence is extremely crude, we here take the coefficients and exponents in the thermodynamic expressions to be free parameters.

The thermodynamic constraints on the free parameters are described in § 2. Next, the important thermodynamic quantities, such as the specific heats and the adiabatic exponents, are derived analytically for a mixture of ideal gas, blackbody radiation, and homogeneous turbulence in § 3. The gas is allowed to have a ratio of specific heats other than 5/3 in order to mimic either a condition of partial ionization or some other nonideal property; illustrations of the thermodynamic effects of turbulence are then presented on the basis of this simple assumption. In § 4, the rigorously correct expressions are worked out for the case in which the gas is

partially ionized. Our main results and some suggestions for potential applications are discussed in § 5.

## 2. THERMODYNAMIC PRELIMINARIES

### 2.1. Constraints

In classical thermodynamics, the thermodynamic identity is given by (Landau & Lifshitz 1958)

$$dQ = T dS = dE + P dV, \quad (6)$$

where  $T$  is temperature,  $S$  is entropy,  $Q$  is quantity of heat,  $E$  is specific internal energy, and  $V = 1/\rho$  is specific volume. Since  $E$  and  $P$  can be expressed in terms of any two independent thermodynamic variables, we choose to write  $E = E(T, V)$  and  $P = P(T, V)$ . Then another well-known thermodynamic formula follows:

$$(\partial E / \partial V)_T = T(\partial P / \partial T)_V - P. \quad (7)$$

Although ionization chemical potentials have been ignored in equation (6) because they are not needed for §§ 2 and 3, they will be included in the more exact calculations of § 4, which explicitly take into account partial ionization of the gas.

If now we utilize the pressure exponents defined by Cox & Giuli (1968),

$$\chi_T = (\partial \ln P / \partial \ln T)_V, \quad \chi_V = (\partial \ln P / \partial \ln V)_T, \quad (8)$$

and if we require for adiabatic processes that

$$E = n' P V, \quad (9)$$

where  $n'$  is a constant, then equation (7) can be rewritten as

$$\chi_V = (\chi_T - 1)/n' - 1. \quad (10)$$

Note that the enthalpy is here just  $H = E + P V = E(n' + 1)/n'$ . The important point to recognize is that in order to satisfy the thermodynamic identity the possible values of the pressure exponents are not free, but are constrained by equation (10).

To take the example of a monatomic ideal gas with gas constant  $R = c_P - c_V$  (which is the difference of specific heat at constant pressure  $c_P$  and specific heat at constant volume  $c_V$ ) one has

$$P_{\text{gas}} = RT/V, \quad E_{\text{gas}} = c_V T, \quad (11)$$

and therefore  $\chi_T = 1$  and  $\chi_V = -1$ , which satisfies equation (10) for any  $n' = c_V/R$ . Likewise, for blackbody radiation with radiation density constant  $a$ , one has

$$P_{\text{rad}} = (1/3)aT^4, \quad E_{\text{rad}} = aT^4 V, \quad (12)$$

which yields  $\chi_T = 4$ ,  $\chi_V = 0$ , and  $n' = 3$ , again satisfying equation (10). These well-known results are no accident, as Joule's experiments on nearly ideal gases originally led to equation (6), while Boltzmann found an expression for  $E_{\text{rad}}$  by integrating equation (7) through the use of the already known equation of state for blackbody radiation (Chandrasekhar 1939).

Since the various pressures in strictly noninteracting systems are additive, equation (10) applies separately to each component of the total pressure of a fluid mixture consisting of gas, radiation, and turbulence in thermodynamic equi-

librium. For the specific case of turbulence, we have

$$P_{\text{turb}} = C v_{\text{turb}}^2 / V, \quad E_{\text{turb}} = \frac{1}{2} v_{\text{turb}}^2. \quad (13)$$

Consequently,  $n' = 1/(2C)$ . Defining the turbulent velocity exponents,

$$\sigma_T = (\partial \ln v_{\text{turb}} / \partial \ln T)_V, \quad \sigma_V = (\partial \ln v_{\text{turb}} / \partial \ln V)_T, \quad (14)$$

we presume that they must satisfy the relation (10), represented here by

$$\sigma_V = C(2\sigma_T - 1), \quad (15)$$

in order for us to be able to treat the turbulent pressure thermodynamically as being part of the scalar total pressure.

If we assume that turbulent elements behave like an ideal gas ( $v_{\text{turb}}^2 \propto T$ ), then  $\sigma_T = 1/2$  and  $\sigma_V = 0$ , with the result that equation (15) holds for any value of  $C$ . Another instructive example applies to a polytropic equation of state,  $P_{\text{turb}} \propto \rho^{\gamma'}$ , in which case  $n' = (\gamma' - 1)^{-1}$ ,  $\sigma_T = 0$ , and  $\sigma_V = -C$ . Thus, for isotropic turbulence, we have  $C = 1/3$ ,  $n' = 3/2$ , and  $\gamma' = 5/3$ , which also represents the polytropic relation for a monatomic ideal gas with a ratio of specific heats  $c_P/c_V = 5/3$ . If, on the other hand,  $C = 1/6$ , we have  $n' = 3$  and  $\gamma' = 4/3$ , which represents, similarly, the case of pure blackbody radiation. Although the case of  $C = 1/6$  violates our initial assumption of isotropy for all the constituents of the medium, we simply invoke the spirit of our crude approximation that the turbulent pressure may be roughly represented thermodynamically as a part of the scalar total pressure. Note that equation (15) cannot be satisfied when either  $P_{\text{turb}}$  or  $v_{\text{turb}}$  is assumed to be constant. More generally, equation (10) shows that a constant pressure is not a thermodynamically viable equation of state for an adiabatic process. (This is not the same thing as holding the pressure constant during an adiabatic process.)

Lastly, since internal energies are also additive, we have

$$E = E_{\text{gas}} + E_{\text{rad}} + E_{\text{turb}}. \quad (16)$$

### 2.2. Specific Heats and Adiabatic Exponents

The thermodynamic quantities of greatest practical use for applications in astrophysics and allied fields are the specific heats and the adiabatic exponents. These are defined by

$$C_V = (dQ/dT)_V, \quad C_P = (dQ/dT)_P, \quad (17)$$

and

$$\Gamma_1 = -(d \ln P / d \ln V)_S, \quad (18)$$

$$\Gamma_2 / (\Gamma_2 - 1) = (d \ln P / d \ln T)_S, \quad (19)$$

$$\Gamma_3 - 1 = -(d \ln T / d \ln V)_S. \quad (20)$$

From the definitions of the three gammas it follows generally that

$$\Gamma_1 / (\Gamma_3 - 1) = \Gamma_2 / (\Gamma_2 - 1). \quad (21)$$

This suggests that we can write the gammas, for expository convenience, as simple ratios:

$$\begin{aligned} \Gamma_1 &= A/D, & \Gamma_2 / (\Gamma_2 - 1) &= A/B, \\ \Gamma_3 - 1 &= B/D. \end{aligned} \quad (22)$$

The notation for the gammas is due to Chandrasekhar (1939), who derived expressions for them in the case of a mixture of ideal gas and radiation. Eddington (1918, 1926), however, was the first author to work out the detailed expressions for  $\Gamma_1$  and  $\Gamma_3$  (called by him  $\gamma$  and  $\gamma'$ ). The gammas appear most often in problems of mechanical instability,  $\Gamma_1$  being associated primarily with dynamical instability,  $\Gamma_2$  with convective instability, and  $\Gamma_3$  with pulsational instability.

### 3. IDEAL GAS, RADIATION, AND TURBULENCE

#### 3.1. Analytic Results

We follow Chandrasekhar (1939) in deriving expressions for the various thermodynamic quantities, here adding a contribution from homogeneous turbulence to the mixture of gas (with a ratio of specific heats  $\gamma = c_P/c_V$ ) and black-body radiation. The total pressure  $P$  is given by equations (4) and (11)–(13), while the total internal energy  $E$  is expressed by equations (11)–(13) and (16).

The turbulent velocity exponents,  $\sigma_T$  and  $\sigma_V$ , appear in the final expressions. Eliminating  $\sigma_V$  with the help of the constraint (15), we find

$$\chi_T = \frac{4 - 3\beta + 2\sigma_T\delta}{1 + \delta}, \quad (23)$$

$$\chi_V = -\frac{\beta + (1 + 2C - 4C\sigma_T)\delta}{1 + \delta}, \quad (24)$$

$$C_V = \frac{c_V}{n\beta} [12 + (n - 12)\beta + C^{-1}\sigma_T\delta], \quad (25)$$

$$C_P - C_V = \frac{c_P - c_V}{n\beta} \left[ \frac{(4 - 3\beta + 2\sigma_T\delta)^2}{\beta + (1 + 2C - 4C\sigma_T)\delta} \right]. \quad (26)$$

Here we have used the definition of the gas polytropic index  $n = (\gamma - 1)^{-1}$  and the two additional definitions

$$\beta = P_{\text{gas}}/(P_{\text{rad}} + P_{\text{gas}}), \quad \delta = P_{\text{turb}}/(P_{\text{rad}} + P_{\text{gas}}). \quad (27)$$

For an ideal monatomic gas,  $c_V = \frac{3}{2}Nk$  and  $c_P = \frac{5}{2}Nk$ , where  $N$  is the number of free particles per unit mass and  $k$  is Boltzmann's constant. If  $\sigma_T$  is positive,  $C_V$  always increases when turbulent pressure and turbulent energy are included, but  $C_P$  may either increase or decrease.

For the adiabatic exponents, we obtain their three constituent factors  $A$ ,  $B$ , and  $D$  as

$$\begin{aligned} A = & 16 - 12\beta + (n - 3)\beta^2 + [(1 + 2C)(12 + n\beta - 12\beta) \\ & + (16 - 12\beta + C^{-1}\beta - 4Cn\beta + 48C\beta - 48C)\sigma_T]\delta \\ & + (1 + 2C)C^{-1}\sigma_T\delta^2, \end{aligned} \quad (28)$$

$$B = (4 - 3\beta + 2\sigma_T\delta)(1 + \delta), \quad (29)$$

$$D = [12 + (n - 12)\beta + C^{-1}\sigma_T\delta](1 + \delta). \quad (30)$$

Notice that in the absence of turbulence, the expressions for the specific heats and for the three gammas reduce to

those of Chandrasekhar. In this case, one finds the limits

$$\text{for } \beta = 1: \quad \Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma, \quad (31)$$

$$\text{for } \beta = 0 \text{ or } \gamma = 4/3: \quad \Gamma_1 = \Gamma_2 = \Gamma_3 = 4/3. \quad (32)$$

Also in this case, the derivatives of all the gammas with respect to  $\beta$  are 0 when  $\gamma = 4/3$ ; accordingly, the gammas increase with increasing  $\beta$  if  $\gamma > 4/3$ , and decrease with increasing  $\beta$  if  $\gamma < 4/3$ .

Several special limiting cases should be noted when turbulence is included, because they also yield very simple expressions for the gammas:

$$\text{for } \beta = 1, C = (\gamma - 1)/2: \quad \Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma, \quad (33)$$

$$\text{for } \gamma = 4/3, C = 1/6: \quad \Gamma_1 = \Gamma_2 = \Gamma_3 = 4/3, \quad (34)$$

$$\text{for } \gamma = 1: \quad \Gamma_1 = (\beta + \delta)/(1 + \delta), \Gamma_2 = 1, \Gamma_3 = 1, \quad (35)$$

$$\text{for } \delta = \infty: \quad \Gamma_1 = \Gamma_2 = \Gamma_3 = 1 + 2C. \quad (36)$$

As a practical matter, the case  $\delta \gg 1$  is not physically very realistic. There must exist some limit on how far the turbulent pressure can exceed gas pressure, if one makes the reasonable assumption that the turbulent velocity cannot greatly surpass the adiabatic (Laplacian) velocity of sound. The latter quantity is given by

$$v_{\text{sound}} = (\partial P / \partial \rho)^{1/2}_S = (\Gamma_1 P / \rho)^{1/2} \quad (37)$$

for nonrelativistic velocities. The condition that  $v_{\text{turb}} \leq v_{\text{sound}}$  imposes the restriction  $P_{\text{turb}}/P \leq C\Gamma_1$  or

$$\delta \leq C\Gamma_1/(1 - C\Gamma_1), \quad (38)$$

where the upper limit on  $\delta$  is clearly of order unity.

If, however, the gas density,  $\rho$ , becomes comparatively low, as it is when the gas is immersed in a strong radiation field,  $v_{\text{sound}}$  may approach the speed of light,  $c$ . In this extremely relativistic case,  $v_{\text{sound}} \rightarrow c/\sqrt{3}$  (Landau & Lifshitz 1959). Since

$$\delta/\beta = P_{\text{turb}}/P_{\text{gas}} = (Cv_{\text{turb}}^2/RT), \quad (39)$$

we can assign a less restrictive upper limit on  $\delta$  for any turbulent velocity:

$$\delta \leq (c^2/3RT)\beta. \quad (40)$$

Although  $\delta$  can apparently grow indefinitely as  $T \rightarrow 0$ , degeneracy must occur at some point in actual practice as the temperature drops.

More realistically, it is unlikely that the turbulent velocity would ever significantly exceed the thermal velocity of the gas. Then, instead of equation (38) or (40), we should have

$$\delta \leq \beta. \quad (41)$$

For later reference, we define  $f = \delta/\beta$  and suppose that  $f \leq 1$ .

Since the turbulent pressure, like the gas pressure, is proportional to the mass density, the two pressures must decline together when  $\rho$  becomes very small. Equations (40) and (41) both show that  $\delta \rightarrow 0$  as  $\beta \rightarrow 0$ .

#### 3.2. Numerical Results

Some numerical examples will now be given. As the three gammas lie close together in value under most conditions,

we choose  $\Gamma_3$  for illustration, following the example of Cox & Giuli (1968). It happens that  $\Gamma_3$  has also the simplest expression,

$$\Gamma_3 = \frac{16(\gamma - 1) + (16 - 15\gamma)\beta + (\gamma - 1)(1 + 2C)C^{-1}\sigma_T\delta}{12(\gamma - 1) + (13 - 12\gamma)\beta + (\gamma - 1)C^{-1}\sigma_T\delta}. \quad (42)$$

First, we note that if  $\sigma_T = 0$  this expression formally reduces to the same expression that holds in the absence of turbulence (although this would not be true for  $\Gamma_1$  and  $\Gamma_2$ ). Second, there exists a critical value of  $\gamma$  above (below) which  $\Gamma_3$  increases (decreases) with increasing  $\beta$ . If we set  $\delta = f\beta$  with  $f$  constant, then

$$\gamma_{\text{crit}} = \frac{4 + (6 - C^{-1})\sigma_T f}{3 + (6 - C^{-1})\sigma_T f}. \quad (43)$$

If, on other hand,  $\delta$  instead of  $f$  is held constant,

$$\gamma_{\text{crit}} = \frac{16 + (26 - 3C^{-1})\sigma_T\delta}{12 + (24 - 3C^{-1})\sigma_T\delta}. \quad (44)$$

Third, there is also a critical value of  $\beta$ , above (below) which  $\Gamma_3$  decreases (increases) with increasing  $\delta$  or  $f$ :

$$\beta_{\text{crit}} = \frac{4(6C - 1)(\gamma - 1)}{3(8C - 1)(\gamma - 1) - 2C}. \quad (45)$$

Since the physically possible values of  $\beta$  fall in the range  $0 \leq \beta \leq 1$ , it is necessary that  $\gamma > 1 + 2C$  in order to have  $\Gamma_3$  decrease. Under most conditions, if  $C = 1/3$ ,  $\Gamma_3$  increases.

Since  $\delta$  must go to 0 as  $\beta$  does, we adopt for illustration the case of constant  $f$ . Figure 1 displays  $\Gamma_3$  as a function of  $\beta$ , based on the use of  $\sigma_T = 1/2$  and  $C = 1/3$ . With these two particular values of the constants,  $\gamma_{\text{crit}} = (8 + 3f)/(6 + 3f)$ . Because  $\beta_{\text{crit}} = 1$  for  $\gamma = 5/3$ ,  $\Gamma_3$  must increase with  $f$  for any realistic ratio of specific heats of the gas ( $\gamma \leq 5/3$ ). Nevertheless, the derived dependence of  $\Gamma_3$  on  $f$  appears to be rather weak and becomes negligible when  $\gamma$

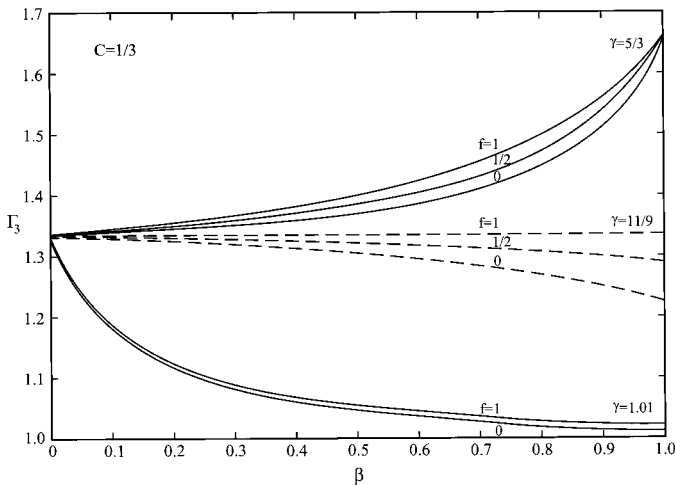


FIG. 1.—Third generalized adiabatic exponent,  $\Gamma_3$ , as a function of the ratio of gas pressure to the sum of gas pressure and radiation pressure,  $\beta$ . The ratio of specific heats of the gas is  $\gamma$ . Turbulent pressure is included with the assigned parameters  $\sigma_T = 1/2$  and  $C = 1/3$  for various strength parameters  $f$ .

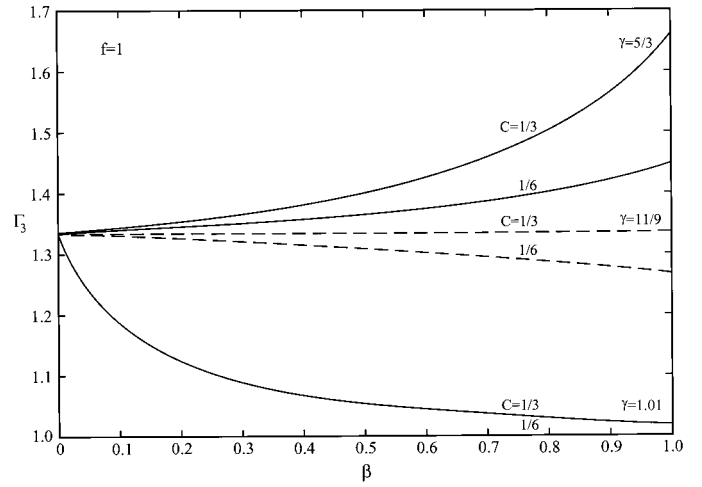


FIG. 2.—Same as Fig. 1, but for  $f = 1$  with two assignments of the anisotropy parameter  $C$ . The isotropic case has  $C = 1/3$ .

drops to very small values. It is also negligible when  $\beta$  is very small.

Figure 2 shows a similar plot, comparing  $C = 1/3$  and  $C = 1/6$  for  $f = 1$ . The reduced value of  $C$  lowers  $\Gamma_3$  everywhere, and has associated with it  $\gamma_{\text{crit}} = 4/3$  for all values of  $f$ , and  $\beta_{\text{crit}} = 0$  for all values of  $\gamma$ . The “true” value of  $C$  is not accurately known, however. Isotropic turbulence implies  $C = 1/3$ , which is adopted in most stellar convection theories (e.g., Gabriel et al. 1975; Kuhfuss 1986; Xiong 1989; Gehmeyer 1992; Yecko, Kolláth, & Buchler 1998). Older studies sometimes assumed  $C = 1/2$  (Unsöld 1955), although Stellingwerf (1982) adopted  $C = 1$ . Henyey, Vardya, & Bodenheimer (1965) recommended  $1 \leq C \leq \pi/2$ ; according to the standard definition of the average turbulent velocity (§ 1), however, it is necessary that  $C \leq 1$ . In Gough’s (1977) theory,  $C$  is a complicated function whose approximate numerical value was not evaluated. The semi-analytic turbulence model of Canuto & Mazzitelli (1991), which approximated the full spectrum of eddy sizes, yields  $C = 0.21$ – $0.23$ . Three-dimensional numerical simulations of solar convection give  $C \approx 1/2$  in the adiabatic region (Li et al. 2002).

#### 4. IONIZING GAS, RADIATION, AND TURBULENCE

There have been many published studies of the thermodynamics of a mixture of ionizing monatomic gas and radiation. The first accurate studies were made by Underhill (1949) and Krishna Swamy (1961). Some of the other investigations have recently been reviewed by Lobel (2001). To include homogeneous turbulence, we here utilize the analytical approach of Cox & Giuli (1968) with the addition of turbulent pressure and turbulent energy as given by equations (13).

Our adopted approach applies, strictly speaking, to a highly diffuse medium in which only one abundant element (hydrogen or helium) is undergoing thermal ionization at a time. This assumption is adequate for most cases of astrophysical interest in which the turbulent pressure becomes of any real significance (e.g., in giant star envelopes or in accretion disks). Next, only two stages of ionization of the ionizing element are treated as being simultaneously in progress. This, too, is nearly always a good assumption.



We define the following four dimensionless quantities, in Cox & Giuli's notation:  $\nu$ , the number of atoms and ions of the ionizing element, divided by the total number,  $N$ , of atoms and ions;  $y$ , the number of atoms of the ionizing element that have lost  $r$  electrons, divided by the total number of atoms and ions of that element;  $\bar{y}$ , the number of free ionization electrons, divided by the total number,  $N$ , of atoms and ions; and the quantity

$$\Xi = \frac{2\bar{y}y(1-y)\nu}{\bar{y}(1+\bar{y}) + y(1-y)\nu} . \quad (46)$$

The ionization potential,  $I$ , is the energy required to remove the  $r$ th electron from the ground state of an  $(r-1)$  times ionized atom. Assuming chemical and thermal equilibrium and neglecting excitation energy, the Saha ionization equation can be used to derive  $y$  from the ionization potential, the partition functions (taken here to be constants), the temperature, and the electron pressure. The thermal kinetic energy per free particle in the ion and electron plasma is taken to be  $\frac{3}{2}kT$ .

With these definitions, we readily find

$$\chi_T = \left[ 4 - 3\beta + \frac{\beta\Xi}{2+\Xi} \left( \frac{3}{2} + \frac{I}{kT} \right) + 2\sigma_T\delta \right] \frac{1}{1+\delta} , \quad (47)$$

$$\chi_V = - \left[ \frac{2\beta}{2+\Xi} + (1 + 2C - 4C\sigma_T)\delta \right] \frac{1}{1+\delta} , \quad (48)$$

$$C_V = \frac{Nk(1+\bar{y})}{\beta} \times \left[ 12 - \frac{21\beta}{2} + \frac{\beta\Xi}{2+\Xi} \left( \frac{3}{2} + \frac{I}{kT} \right)^2 + C^{-1}\sigma_T\delta \right] , \quad (49)$$

$$C_P - C_V = -(\Gamma_1/\chi_V + 1)C_V . \quad (50)$$

After considerably more algebra, we get also

$$\begin{aligned} A = & 32 - 24\beta - 3\beta^2 + \beta^2\Xi \left[ 4 \left( \frac{1-\beta}{\beta} \right) + \left( \frac{5}{2} + \frac{I}{kT} \right) \right]^2 \\ & + 2\sigma_T\delta \left\{ 16 + \beta(C^{-1} - 12) + 2\beta\Xi \left[ \frac{4}{\beta} + \left( \frac{I}{kT} - \frac{3}{2} \right) \right] \right\} \\ & + (1 + 2C - 4C\sigma_T)\delta \left\{ 3(8 - 7\beta) \right. \\ & \left. + \beta\Xi \left[ 12 \left( \frac{1-\beta}{\beta} \right) + \left( \frac{3}{2} + \frac{I}{kT} \right) \left( \frac{5}{2} + \frac{I}{kT} \right) - \frac{I}{kT} \right] \right\} \\ & + (1 + 2C)C^{-1}\sigma_T\delta^2(2 + \Xi) , \end{aligned} \quad (51)$$

$$\begin{aligned} B = & \left\{ 2(4 - 3\beta) + \beta\Xi \left[ 4 \left( \frac{1-\beta}{\beta} \right) + \left( \frac{5}{2} + \frac{I}{kT} \right) \right] \right. \\ & \left. + 2(2 + \Xi)\sigma_T\delta \right\} (1 + \delta) , \end{aligned} \quad (52)$$

$$\begin{aligned} D = & \left\{ 3(8 - 7\beta) + \beta\Xi \right. \\ & \times \left[ 12 \left( \frac{1-\beta}{\beta} \right) + \left( \frac{3}{2} + \frac{I}{kT} \right) \left( \frac{5}{2} + \frac{I}{kT} \right) - \frac{I}{kT} \right] \\ & \left. + C^{-1}\sigma_T\delta(2 + \Xi) \right\} (1 + \delta) . \end{aligned} \quad (53)$$

These expressions reduce to those of Cox & Giuli (1968) for the case of no turbulence ( $\delta = 0$ ). In the case of a neutral or a fully ionized gas ( $\Xi = 0$ ), the expressions we derived in § 3.1 are recovered if in those expressions  $\gamma$  is set to  $5/3$  and  $N$  is identified as the total number of free particles per unit mass. Cox & Giuli have shown, in general, that  $0 \leq \Xi \leq 1/2$ . Partial ionization always lowers the three gammas in a dramatic way that is approximately mimicked by taking  $\gamma < 5/3$  in the expressions for the fully neutral (or the fully ionized) case.

## 5. DISCUSSION

What are the conditions under which the turbulent pressure may be treated, at least formally, as a thermodynamic variable? We have seen that a certain relation among the constants  $C$ ,  $\sigma_T$ , and  $\sigma_V$  must be satisfied (eq. [15]). Physically, can this condition ever be realized? If turbulence is fast and homogenous, it probably behaves much like an ideal gas, the kinetic energy of the turbulent eddies being linearly combinable with the thermal kinetic energy of the gas particles of which the eddies are composed. Thus,  $E = \frac{1}{2}(v_{\text{gas}}^2 + v_{\text{turb}}^2)$ . Therefore, from a thermodynamic point of view, we would expect our approach to improve as an approximation to the true situation if  $v_{\text{turb}}$  is of the order of  $v_{\text{gas}}$  and if the turbulent eddies are small and isotropic. Of course, a problem remains of the proper choices of  $C$  and  $\sigma_T$ . Plausible choices, such as  $\sigma_T = 1/2$  or  $0$ , with  $C = 1/3$ , lead to results that do not greatly differ from each other.

If, however, the turbulent timescale is long and the main flux-carrying eddies are large, anisotropy of the turbulence would doubtless also be large. Our approach would then break down for a number of reasons. In this extreme case, the turbulent pressure can only react very slowly to rapid expansions and contractions of the gas. If the turbulent velocity or turbulent pressure remains approximately constant in time, the lifting effect of the turbulent pressure gradient would act formally to reduce the effective gravity rather than to modify the spatial gradient of gas pressure and radiation pressure, and so equation (3) should then be reorganized to read:  $dp/dz = -g\rho - dP_{\text{turb}}/dz$ . The turbulent pressure term  $dP_{\text{turb}}/dz$  would therefore have to be treated hydrodynamically in some way, or else one might simply set  $v_{\text{turb}}$  or  $P_{\text{turb}}$  to be constant in time. In either case, standard thermodynamics could still apply to the gas and radiation fields, and so the traditional forms of the gammas would remain the relevant ones.

In giant star envelopes, turbulent velocities in the ionization zones attain or even exceed sound speed. The approximate effect of turbulence on the envelope structure may then be treated by the methods outlined in this paper. In cool giants,  $\beta \approx 1$  and if  $C \approx 1/3$  turbulence would not be expected to have a large effect in layers well outside the ionization zones of hydrogen and helium as far as the stability criteria (which depend on the three gammas) are concerned. Deep inside the ionization zones where the three gammas lie close to 1, turbulence again should be thermodynamically (but not hydrodynamically) ineffectual. Its overall influence, therefore, *might* be relatively slight in cool giants. In hot supergiant stars,  $\beta \approx 0$  and, because of this circumstance, turbulence ought not to have a large influence on the stability properties of any layers of these stars. Obviously, detailed numerical calculations need to be made in order to check these predictions, especially since the turbulent

pressure must affect the star's equilibrium structure through the equation of state and the equation of hydrostatic equilibrium. In hot supergiant stars, however, the low value of  $\beta$  means that the structure is already largely determined by radiation pressure; therefore, any extra contribution from turbulent pressure is unlikely to amount to much. This would not necessarily be the case for cool giants, however (Stellingwerf 1976; Jiang & Huang 1997).

Jiang & Huang (1997) and Huang & Yu (1998) also have considered the effects of turbulence on thermodynamic quantities. However, they have adopted the extreme (and inconsistent) values  $C = 1$  and  $n' = 3$ , and have ignored the constraint given by equation (10) that must be fulfilled in order to satisfy the thermodynamic identity. In addition, they have calculated only one of the three gammas and only one of the two specific heats.

An alternative approach to turbulence that has been utilized by Li et al. (2002) is to treat it in the manner of a mag-

netic field, so that the turbulent energy per unit mass becomes a new state variable that is abruptly perturbed from one constant value to another. In this case, the thermodynamic identity changes to a nonstandard form, differing from equation (6) by displaying  $p$  instead of  $P$ . The results that follow would obviously differ from our present ones. It might even be possible to treat turbulence by using a chemical potential. However, we believe that our present results can illustrate, roughly, the magnitude and direction of changes in thermodynamic quantities that turbulence causes under specified conditions.

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